ELSEVIER
Available online at www.sciencedirect.com
SciVerse ScienceDirect
Physics Procedia 25 (2012) 1492 – 1499
Physics Procedia

2012 International Conference on Solid State Devices and Materials Science

The Noval Properties and Construction of Multi-scale Matrix-valued Bivariate Wavelet wraps

ZHANG Hai-mo

Department of Math. Science,Huanghuai University,Zhumadian 463000, China

Abstract

In this paper, we introduce matrix-valued multi-resolution structure and matrix-valued bivariate wavelet wraps. A constructive method of semi-orthogonal matrix-valued bivari-ate wavelet wraps is presented. Their properties have been characterized by using time-frequency analysis method, unitary extension principle and operator theory. The direct decom-position relation is obtained.

Keywords :bivariate, iterative method, matrix-valued mul-tiresolution structure, unitary extension principle, wavelet wraps

1.Introduction

Wavelet analysis has become a popular subject in scient-ific research for twenty years. It has been a powerful tool for exploring and solving many complicated problems in natural science and engineering computation. Sampling theorems play a basic role in digital signal processing. They ensure that continnuous signals can be represented and processed by their discrete samples.The classical Shannon Sampling Theorem a-sserts that bandlimmited signals can be exactly represented by their uniform samples as long as the sampling rate is not less than the Nyquist rate. This theorem has been proved to be fundamental in many applications of signal processing and comm-unication theory. When we use the wavelet decompositions in digital signal pro-cessing, the coefficients in the high level re-presentation can be chosen tobe samples of the continuous signal. However, for the wavelet decomposition, the Mallat alg-orithm is often used, and FIR filters are preferred, although fi-lters with exponentially decaying impulse responses also provide satisfactory results for many applications. Construction of wavelet bases is an important aspect of wavelet analysis, and multiresolution analysis method is one of importment ways of designing various wavelet bases. The main traits of the wavelet trans-form is to hierarchically decompose

1875-3892 © 2012 Published by Elsevier B.V. Selection and/or peer-review under responsibility of Garry Lee
Open access under CC BY-NC-ND license. doi:10.1016/j.phpro.2012.03.267

general functions, as a signal or a process, into a set of approximation functions wi-th different scales. Recently, Multiwavelets [1,2] have been a-pplied to many aspects in technology and science,such as,image compress, signal processing [3], solving Integral Equations [4] and so on, mainly because of their ability to offer properties like symmetry, orthogonality, short support at the same time.It is noticed that multiwavelets can be generated from the compo-nent function in multiple vector-valued wavelets.

Researching into multiple vector-valued wavelet is useful in multiwavelet th-eory. Chen [5] introduced the notion of multiple vector-valued wavelets and studied the existence and construction of orthog-onal multiple vector-valued wavelets. Fowler and Li [6] imple-mented orthogonal multiple vector-valued wavelet transforms to study fluid flows in oceanography and aerodynamics. However, multiwavelets and multiple vector-valued wavelets are di-fferent in the following sense. Prefiltering is usually required for discrete multi-wavelet ransforms but not necessary for disc-rete multiple vector-valued wavelet transforms. Therefore it is necessary to study multiple vector-valued wavelets. However, as yet there has not been a general method to obtain orthogon-al multiple vector-valued wavelets. In the signal denosing met-hod, wavelet ana-lysis is a new analytical tool which develops on the basis of the Fourier analysis.The location and multiscale resolution function simultaneously contained in time domain and frequency domain. In order to enhance the application of wavelet analysis in the signal process-ing, and to improve the accuracy of signal processing as far as possible, predecessors have done a large number of practices and explorations, resul-ting in a lot of signal processing methods based on wavelet tra-nsform. They ensure that conti-nuous signals can be represent-ed and processed by their discrete samples.We shall present an algorithm for designing a sort of finitely supported orthogonal multiple vector-valued scaling functions and wavelets. We also study the traits of multiple vector-valued wavelet packs.

2.Matrix-valued multiresolution analysis

The multiple vector-valued multiresolution analysis is in--troduced and the definition for orthogonal multiple vector-va-lued wavelets is given. Moreover, we are now ready to discuss the construction of orthogonal multiple vector -valued wavele--ts. Let $\mathbb{C}$ and $\mathbb{R}$ denote all complex and all real numbers,resp-
ectively. $\mathbb{Z}$ and $\mathbb{Z}_+$ stand for, respectively, all inters and nonn-egative integers. Set $s,a\in\mathbb{Z}$ be a constant and $s,a\geq 2$. The signal space $L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ is defined to be the set of all multiple vector-valued functions $\Phi(y)$, i.e.

$$\Phi(y)=\begin{pmatrix}\phi_{11}(y) & \phi_{12}(y) & \cdots & \phi_{1s}(y)\\ \phi_{21}(y) & \phi_{22}(y) & \cdots & \phi_{2s}(y)\\ \cdots & \cdots & \cdots & \cdots\\ \phi_{s1}(y) & \phi_{s2}(y) & \cdots & \phi_{ss}(y)\end{pmatrix},$$

where $\phi_{j,l}(y)\in L^2(\mathbb{R}^2)$, $j,l=1,2,\cdots s$. Examples of matrix-valued signals are video images in which $\phi_{j,l}(y)$ denotes the pixel at the time t the j th row and the l th column. For $\Phi(y)\in L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$, $\|\Phi\|=\sqrt{\sum_{j,k=1}^{s}\int_{\mathbb{R}^2}|\phi_{j,k}(y)|^2\,dy}$, and the intgration of multiple vector-valued function $\Phi(y)$ is defined to be

$$\int_{\mathbb{R}^2}\Phi(y)\,dy:=\Big(\int_{\mathbb{R}^2}\phi_{j,l}(y)dy\Big)_{j,l=1}^{s} \qquad (1)$$

i.e., the matrix of the integral of every scalar functions $\phi_{j,l}(x)$, $j,l=1,2,\cdots,s$.
For arbitrary $\Upsilon,\Gamma \in L^2(\mathbb{R},\mathbb{C}^{s\times s})$, their sy-mbol inner product is defined by

$$\left\langle \Upsilon,\Gamma \right\rangle := \int_{\mathbb{R}^2} \Upsilon(y)\cdot\Gamma(y)^* dy , \qquad (2)$$

where $*$ means the transpose and the complex conjugate.

Definition 1. A family of mul-tiple vector -valued function $\{\Phi_u(y)\}_{u\in\mathbb{Z}^2} \subset L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ is called an orthonormal basis, if the following condition is satisfied:

$$\left\langle \Phi_j,\Phi_l \right\rangle := \delta_{j,l} I_s \qquad j,l\in\mathbb{Z}^2 , \qquad (3)$$

and $\Upsilon(x) \in L^2(\mathbb{R},\mathbb{C}^{s\times s})$, there is a sequence of $s\times s$ constant matrice M_u such that

$$\Upsilon(y) = \sum_{u\in\mathbb{Z}^2} M_u \Phi_u(y) \ , \qquad (4)$$

where I_s denotes the $s\times s$ identity matrix and $\delta_{j,l}=1$ when $j=l$ and $\delta_{j,l}=0$ when $j\neq l$.

Definition 2. A binary multiple vector-valued multiresolution analysis of $L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ is a nested sequence of closed subsp-aces U_j , $j\in Z$ of $L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ such that it follows

(i) $U_j \subset U_{j+1}, \forall\, j\in Z;\ a\in Z \quad a\geq 2$.

(ii) $\Phi(y)\in U_0 \Leftrightarrow \Phi(a^j y)\in U_j, \forall\, j\in Z;$

(iii) $\bigcap_{j\in Z} U_j = \{O\};\ \bigcup_{j\in Z} U_j$ is dense in $L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$

(iv) There is a $F(t)\in X_0$ such that its translates $\{H_v(y):=H(y-v),\ v\in Z^2\}$ form a *Riesz* basis for U_0.

Since $H(t)\in U_0 \subset U_1$, by definition 1 and (2), there is a finite-term sequence of $s\times s$ matrice $\{P_v\}_{v\in\mathbb{Z}^2}$ such that

$$H(y) = a^2 \cdot \sum_{v\in\mathbb{Z}^2} P_v H(ay-v). \qquad (5)$$

Equation (5) is called a refinement equation and $H(y)$ is cal-led a vector scaling functions.

Let $W_j, j\in\mathbb{Z}$, denote the complementary subspace of U_j in U_{j+1} and there exist a^2-1 multiple vector-valued functions $\Psi_\iota(t)\in L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ such that $G_{\iota,j,k}(x)=a^{j/2}G_\iota(a^j x-k),\ j,k\in\mathbb{Z}^2$ forms a *Riesz* basis of W_j, where $\iota\in\Lambda=\{1,2,\cdots a^2-1\}$. It is clear that $G_\iota(y)\in W_0 \subset U_1$ .Hence there exists a sequence of $s\times s$ matrices $\{B_k^\iota\}_{k\in Z^2}$ $\iota\in\Lambda=\{1,2,\cdots a^2-1\}$, such that

$$G_\iota(t) = a^2 \cdot \sum_{k\in\mathbb{Z}^2} B_v^\iota H(ay-k). \qquad (6)$$

We call $H(x)$ an orthogonal multiple vector-valued sca-ling functions, if it satisfies

$$< H(\cdot), H(\cdot-k) >= \delta_{0,k} I_s \qquad k\in\mathbb{Z}^2. \qquad (7)$$

We say that $G_\iota(y)\in L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ are orthogonal multiple vector-valued wavelets associated with orthogonal multiple vector-valued scaling functions $H(y)$, if

$$< H(\cdot), G_\iota(\cdot-n) >= O, \quad n\in\mathbb{Z}^2, \iota\in\Lambda; \qquad (8)$$

$$< G_\iota(\cdot), G_v(\cdot-n) >= \delta_{\iota,v}\delta_{0,n} I_s,\ n\in\mathbb{Z}^2, \iota,v\in\Lambda. \quad (9)$$

Then, we have the following results from (4)-(7).

Theorem 1. Suppose that $H(y)$ defined by (5), is an ortho-gonal multiple vector -valued scaling functions. Then, for any $u\in Z^2$, we have

$$a^2 \sum_{k\in\mathbb{Z}^2} P_{k+au}(P_k)^* = \delta_{0,u} I_s. \qquad (10)$$

Proof. Substituting equations (5) into the orthogonality relat-ion formula (7), we have

$$\delta_{0,u} I_s =< H(\cdot), H(\cdot - k) >$$
$$= a^4 \cdot \sum_{j,k\in\mathbb{Z}^2} \int_{R^2} P_j H(ay-j) H(ay-ak-n)^* (P_n)^* dt$$
$$= a^2 \cdot \sum_{j,k\in\mathbb{Z}^2} P_j < H(\cdot - j), H(\cdot - au - k) > (P_k)^*$$
$$= a^2 \cdot \sum_{n\in\mathbb{Z}^2} P_{n+ak} (P_n)^*$$

Theorem 2[2]**.** Assume that $G_t(y)$ are multiple vector-valued functions in $L^2(\mathbb{R}^2, \mathbb{C}^{s\times s})$. Then $G_t(y)$ are orthogonal multi-ple vector-valued wavelet functions associated with orthogo-nal vector-valued scaling functions $H(y)$, then we have

$$\sum_{k\in Z^2} P_{k+mu} (\tilde{B}_k)^* = O, \quad u \in Z^2, \tag{11}$$

$$a^2 \sum_{k\in\mathbb{Z}^2} B_{k+an} (\tilde{B}_k)^* = \delta_{0,n} I_s, \quad n \in \mathbb{Z}^2. \tag{12}$$

Thus, both Theorem 2 and (11)-(12) provided an approach to design orthogonal multiple vector-valued wavelets with $a = 2$.

3.Construction of vector-valued wavelets

In what follows, we investigate the construction of compactly supported orthogonal matrix-valued wavelet functions a--nd present an algorithm for constructing them **Theorem 3**[7] **.** Let $H(y) \in L^2(\mathbb{R}^2, \mathbb{C}^{s\times s})$ be a 5-cocfficient compactly supported orthogonal multiple vector-valued scaling function saisfying the following equation:

$$H(t) = 4P_0 H(2y) + 4P_1 H(2y-1) + \cdots + 4P_4 H(2y-4). \tag{13}$$

Assume that there exists an integer $v, 0 \le v \le 2$,such that the matrix M defined in the following equation, is not only an in--vertible matrix but also a Hermitian matrix:

$$M^2 = ((1/4) I_s - P_v P_v^*)^{-1} P_v P_v^* . \tag{14}$$

Define

$$\begin{cases} B_j = M P_j, & j \ne v, \\ B_j = -M^{-1} P_j, & j = v, \end{cases} \quad j, v \in \{0,1,2,3,4\}. \tag{15}$$

Then $G(y)$, defined in (17), is an orthogonal matrix-valued wavelets associated with $H(y)$:

$$G(y) = 4\{B_0 H(2y) + B_1 H(2y-1) + \cdots + B_4 H(2y-4)\}. \tag{16}$$

4.The traits of a sort of wavelet wraps

We will consider wavelet wraps in $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, we set

$$\Psi_0(y) = H(y),\ \Psi_1(y) = G(y), \Omega_v^{(0)} = P_v, \Omega_v^{(1)} = B_v, v \in \mathbb{Z}^2 .$$

Definition 3. A set $\{\Psi_{2n+\tau}(y), n=0,1,2,\cdots,\tau=0,1,2,3\}$ is called multiple vector-valued wavelet packs concerning the orthogonal multiple vector -valued scaling functions $H(y)$, where

$$\Psi_{2n+\tau}(y)=4\cdot\sum_{k\in\mathbb{Z}^2}\Omega_k^{(\tau)}\Psi_n(y),\tau=0,1,2,3. \quad (17)$$

Taking the Fourier transform for the both side of (17) gives

$$\widehat{\Psi}_{2n+\tau}(2\omega)=\underline{\Omega}^{(\tau)}(\omega)\widehat{\Psi}_n(\omega),\ \omega\in\mathbb{R},\ \tau=0,1,2,3, \quad (18)$$

$$\underline{\Omega}^{(\tau)}(\omega)=\sum_{k\in\mathbb{Z}^2}\Omega_k^{(\tau)}\cdot\exp\{-ik\omega\},\ \omega\in\mathbb{R}. \quad (19)$$

Lemma 1[6]. Assume $\Psi_\mu(y)\in L^2(\mathbb{R}^2,\mathbb{C}^{s\times s})$ $\mu\in\Gamma$ are orthogonal multiple vector-valued wavelet functions associated with $\Psi_{\underline{0}}(y)$. Then, for $\mu,\nu\in\Gamma_0$, we have

$$\sum_{\rho\in\Gamma_0}\underline{\Omega}^{(\mu)}((\omega+2\rho\pi)/a)\underline{\widetilde{\Omega}}^{(\nu)}((\omega+2\rho\pi)/a)^*=\delta_{\mu,\nu}I_s. \quad (20)$$

Theorem 4[6]. Assume $\{\Psi_n(y), n=0,1,2,\cdots\}$ is multiple vector-valued wavelet wraps concerning the orthogonal multiple vector -valued wraps scaling functions $H(y)$,Then

$$\left\langle\Psi_{2n}(\cdot),\Psi_{2n+1}(\cdot-k)\right\rangle=O,\quad n\in\mathbb{Z}_+,k\in\mathbb{Z}. \quad (21)$$

Theorem 5. If $\{\Psi_\ell(y),\ell=0,1,2,\cdots\}$ is a multiple vector -valued wavelet wraps concerning the orthogonal multiple vec- tor -valued scaling functions $H(y)$, then, we have

$$\left\langle\Psi_m(\cdot),\Psi_n(\cdot-k)\right\rangle=\delta_{m,n}\delta_{0,k}I_s, m,n\in\mathbb{Z}_+\bigcup\{0\},\ k\in\mathbb{Z}. \quad (22)$$

Proof. Formula (22) holds for $m=n$ by Theorem 1. For the case of $m\neq n$, without loss of generality, we assume $m>n$. We can write m,n as $m=2[m/2]+\tau_1$, $n=2[n/2]+\rho_1$, where $\tau_1,\rho_1=0,1$. (i) If $[m/2]=[n/2]$, then $\tau_1\neq\rho_1$. By (19), (20) and (22), we get that

$$2\pi\left\langle\Psi_m(\cdot),\Psi_n(\cdot-k)\right\rangle$$
$$=\int_{\mathbb{R}}\underline{\Omega}^{(\tau_1)}(\omega/2)\widehat{\Psi}_{[m/2]}(\omega/2)\widehat{\Psi}_{[n/2]}(\omega/2)^*$$
$$\cdot\underline{\Omega}^{(\rho_1)}(\omega/2)^*\cdot e^{ik\omega}\,d\omega=O.$$

(ii) For $[m/2]\neq[n/2]$. Let $[m/2]=2[[m/2]/2]+\tau_2$, $[n/2]=2[[n/2]/2]+\rho_2$,where $\tau_2,\rho_2=0,1$.

$$2\pi\left\langle\Psi_m(\cdot),\Psi_n(\cdot-k)\right\rangle$$
$$=\int_0^{2^{\kappa+1}\pi}\prod_{\sigma=1}^{\kappa}\underline{\Omega}^{(\tau_\sigma)}(\omega/2^\sigma)\cdot O\cdot\prod_{\sigma=1}^{\kappa}\underline{\Omega}^{(\rho_\sigma)}(\omega/2^\sigma)^\dagger=O$$

Theorem 5. If $\{\Psi_\alpha(y),\alpha\in\mathbb{Z}_+^2\}$ and $\{\widetilde{\Psi}_\alpha(y),\alpha\in\mathbb{Z}_+^2\}$ are biorthogonal ninary wavelet packss according to concerning to a pair of biorthogonal vector-valued scaling functions $\Psi_0(y)$ and $\widetilde{\Psi}_0(y)$, then for any $a=4$, $\alpha,\sigma\in\mathbb{Z}_+^2$, we have

$$\left\langle\Psi_\alpha(\cdot),\widetilde{\Psi}_\sigma(\cdot-k)\right\rangle=\delta_{\alpha,\sigma}\delta_{\underline{0},k}I_s,\ k\in\mathbb{Z}^2. \quad (23)$$

Proof. When $\alpha=\sigma$, (28) follows by Lemma 1. as $\alpha\neq\sigma$ and $\alpha,\sigma\in\Gamma_0$, it follows from Lemma 2 that (28) holds, too. Assuming that α is not equal to β, as well as at least one of $\{\alpha,\sigma\}$ doesn't belong to Γ_0, we rewrite α,σ as $\alpha=4\alpha_1+\ \rho_1,\ \ \sigma=4\sigma_1+\mu_1$, where $a=4,\rho_1,\mu_1\in\Gamma_0$. **Case 1**. If $\alpha_1=\sigma_1$, then $\rho_1\neq\mu_1$. (23) follows by virtue of (17) as well as Lemma 1, i.e.,

$$(2\pi)^s\ \left\langle\Psi_\alpha(\cdot),\widetilde{\Psi}_\sigma(\cdot-k)\right\rangle$$

$$=\int_{R^2}\widehat{\Psi}_{4\alpha_1+\rho_1}(\omega)\widehat{\widetilde{\Psi}}_{4\sigma_1+\mu_1}(\omega)^*\cdot\exp\{ik\cdot\omega\}d\omega$$

$$=\int_{[0,2\pi]^2}\delta_{\rho_1,\mu_1}I_s\cdot\exp\{ik\cdot\omega\}d\omega=O.$$

Case 2 If $\alpha_1\neq\sigma_1$, order $\alpha_1=4\alpha_2+\rho_2,\quad \sigma_1=4\sigma_2+\mu_2$, where $\alpha_2,\sigma_2\in Z_+^s$, and $\rho_2,\mu_2\in\Gamma_0$. Provided that $\alpha_2=\sigma_2$, then $\rho_2\neq\mu_2$. Similar to Case 1, (28) can be established. When $\alpha_2\neq\sigma_2$, order $\alpha_2=4\alpha_3+\rho_3,\sigma_2=4\sigma_3+\mu_3$, where $\alpha_3,\sigma_3\in Z_+^s,\rho_3,\mu_3\in\Gamma_0$. Thus, after taking finite steps (de-noted by κ), we obtain $\alpha_\kappa\in\Gamma_0$, and $\rho_\kappa,\mu_\kappa\in\Gamma_0$. If $\alpha_\kappa=\ \sigma_\kappa$,then $\rho_\kappa\neq\mu_\kappa$. (28) holds. If $\alpha_\kappa\neq\sigma_\kappa$, then we obtain

$$\left\langle\Psi_\alpha(\cdot),\widetilde{\Psi}_\sigma(\cdot-k)\right\rangle=\frac{1}{(2\pi)^2}\int_{R^2}\widehat{\Psi}_\alpha(\omega)\widehat{\widetilde{\Psi}}_{\sigma_1}(\omega)^*\cdot e^{ik\cdot\omega}d\omega$$

$$=\frac{1}{(2\pi)^2}\int_{R^2}\underline{\Omega}^{(\rho_1)}(\omega/4)\underline{\Omega}^{(\rho_2)}(\omega/16)\widehat{\Psi}_{\alpha_2}(\omega/16)$$

$$\cdot\widehat{\widetilde{\Psi}}_{\sigma_2}(\omega/16)^*\widetilde{\underline{\Omega}}^{(\mu_2)}(\omega/16)^*\widetilde{\underline{\Omega}}^{(\mu_1)}(\omega/4)^*\cdot e^{ik\cdot\omega}d\omega=\cdots\cdots$$

$$=\frac{1}{(2\pi)^2}\int_{([0,2\cdot4^\kappa\pi]^2}\{\prod_{l=1}^{\kappa}\Omega^{(\rho_l)}(\omega/4^l)\}\cdot O$$

$$\cdot\{\prod_{l=1}^{\kappa}\widetilde{\underline{\Omega}}^{(\mu_l)}(\frac{\omega}{4^l})\}^*\cdot\exp\{-ik\cdot\omega\}d\omega=O\cdot$$

Therefore, for any $\alpha,\sigma\in\mathbb{Z}_+^2$, (28) is established. **Theorem 6**. Let $h(y)$, $\tilde{h}(y)$, $\psi_\iota(y)$ and $\widetilde{\psi}_\iota(y)$, $\iota\in\Lambda$ be functions in $L^2(R^2)$ defined by (5), (6), (9) and (10), respectively. Assume that conditions in Theorem 1 are satisfied. Then, for any function $f(y)\in L^2(R^2)$, and any integer n,

$$\sum_{k\in Z^2}\left\langle f,\tilde{h}_{n,k}\right\rangle h_{n,k}(y)=\sum_{\iota=1}^{4}\sum_{v=-\infty}^{n-1}\sum_{k\in Z^2}\left\langle f,\tilde{\psi}_{\iota:v,k}\right\rangle\psi_{\iota:v,k}(y). \quad (24)$$

Furthermore, for any $f(y)\in L^2(R^2)$,

$$f(y)=\sum_{\iota=1}^{4}\sum_{v=-\infty}^{\infty}\sum_{k\in Z^2}\left\langle f,\widetilde{\psi}_{\iota:v,k}\right\rangle\psi_{\iota:v,k}(y). \qquad (25)$$

Proof. (i) Consider, for $\sigma\geq0$, $\sigma\in Z$, the operator $E_\sigma: L^2(R^2)\to L^2(R^2)$ such that

$$E_\sigma\Upsilon(y)\equiv\Upsilon_\sigma(y)\equiv\sum_{k\in Z^2}\left\langle\Upsilon,\tilde{h}_{(-\sigma),k}\right\rangle h_{(-\sigma),k}(y).$$

Then the operator E_σ are well defined and uniformly bounded in the norm on $L^2(R^2)$. To show that

$E_\sigma \to 0$ as $\sigma \to \infty$, it is sufficient to show that, for all $g(s)$ in any dense subspace of band-limited functions in $L^2(R^6)$,

$$\sum_{k\in Z^2} \left\langle g, \tilde{h}_{(-\sigma),k} \right\rangle h_{(-\sigma),k} \to 0 \text{ as } \sigma \to \infty.$$

In particular, we may choose the dense set of functions $g(s)$, whose Fourier transform have compact support, is continuous, and vanishes in a neighborhood of 0.

$$\left\| \sum_k \left\langle g, \tilde{h}_{(-\sigma),k} \right\rangle h_{(-\sigma),k} \right\| \le B^{1/2} \left(\sum_k \left| \left\langle g, \tilde{h}_{(-\sigma),k} \right\rangle \right|^2 \right)^{1/2} \quad (26)$$

where B is the Bessel bound of $\{h_{-\sigma,k}\}_k$. Implementing standard calculation of the right-hand side of (13), we have

$$\sum_k \left| \left\langle g, \tilde{h}_{(-\sigma),k} \right\rangle \right|^2$$

$$= \int \sum_u \hat{g}\left(\omega + 2^{-\sigma} u\right) \overline{\hat{\tilde{h}}_v\left(2^\sigma \omega + u\right)} \hat{\tilde{h}}_v\left(4^\sigma \omega\right) \overline{\hat{g}(\omega)} d\omega$$

$$\le \left(B^*\right)^{1/2} \int (4^{-\sigma} \sum_u |\hat{g}(\omega + 4^{-\sigma} u)^2 |)^{1/2} \cdot 4^\sigma \left| \hat{\tilde{h}}_v\left(4^\sigma \omega\right) \hat{g}(\omega) \right| d\omega$$

where B^* is the Bessel bound of $\{\tilde{h}_{-\sigma,k}\}_k$. Following the lead of [8] and since $\hat{f}(\omega)$ is continuous with compact support, the term $4^{-\sigma} \sum_n \left(\left| \hat{g}\left(\omega + 4^{-\sigma} n\right) \right|^2 \right)^{1/2} \le C^2 < +\infty$, being a Riemann sum to the finite integral $\int \left| \hat{g}(\omega + t) \right| dt$. Moreover, since $\hat{g}(\omega)$ vanishes in a neighborhood of 0 for all $\|\omega\| < \delta_g$, we get that

$$\sum_k \left| \left\langle g, \tilde{h}_{-\sigma,k} \right\rangle \right|^2 \le (B^*)^{1/2} C \int | 4^\sigma \hat{\tilde{h}}(4^\sigma \omega) \hat{g}(\omega) | d\omega$$

$$\le \left(B^*\right)^{1/2} C \|g\|_2 \left(\int_{\|\omega\| \ge \delta_g} | 4^\sigma \hat{\tilde{h}}(4^\sigma \omega) |^2 d\omega \right)^{1/2}$$

Note that the last integral at the right-hand side tends to 0 as $\sigma \to +\infty$. This proves the first part of the theorem since, by using (18) recursively, we have

$$\lambda_\sigma(y) = \sum_{k\in Z^2} \left\langle \lambda, \tilde{h}_{n,k} \right\rangle h_{n,k}(y) - \sum_{\iota=1}^{4} \sum_{v=-\infty}^{n-1} \sum_{k\in Z^2} \left\langle \lambda, \tilde{\psi}_{\iota:v,k} \right\rangle \psi_{\iota:v,k}(y).$$

(ii) Since $\overline{\bigcup V_\ell} = L^2(R^2)$, for any $\hbar \in L^2(R^2)$ and any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) > 0$, and for any $n > n_0$ $\exists$ $g \in V_{n_0} \subset V_n$ such that $g(s) = \sum_{k\in Z^2} \left\langle g, \tilde{h}_{v,k} \right\rangle h_{v,k}(s)$ Moreover, for $C = \sqrt{BB^*}$, $\|\Upsilon - g\|_2 < (1+C)^{-1} \varepsilon$. Now, by (11), for all $n > n_0$, we have

$$\left\| \Upsilon - \sum_{\iota=1}^{4} \sum_{v=-\infty}^{n-1} \sum_{k\in Z^2} \left\langle \Upsilon, \tilde{\psi}_{\iota:v,k} \right\rangle \psi_{\iota:v,k} \right\|_2$$

$$\leq \left\|\Upsilon - g\right\|_2 + C\left\|\Upsilon - g\right\|_2 = \left\|\Upsilon - g\right\|_2 \left(1 + C\right) < \varepsilon .$$

References

[1] L. Telesca, et al.. “Multiresolution wavelet analysis of earthquakes,” Chaos, Solitons & Fractals, Vol.22,No.3, PP. 741-748, 2004.

[2] H. He, S. Cheng. “Home network power-line communi-cation signal processing based on wavelet packet analysis,” IEEE Trans Power Delivery, Vol.20, No.3, PP. 1879-1885.,2005.

[3] N. Zhang, X.Wu X. Lossless Compression of Color Mosaic Images,” IEEE Trans Image processing Vol. 15, No. 16, PP.1379-1388, 2006.

[4] Q. Chen , A . Huo. “The research of a class of biorth-ogonal compactly supported vector-valued wavelets,” Chaos , Solitons & Fractals , 41 (2), pp . 951-961, August 2009”

[5] X. G. Xia, B. W.Suter. “Vector-valued wavelets and vector filter banks,” IEEE Trans. signal Processing, 1996,44(10): 508-518.

[6] Q. Chen, etal. “A study on compactly supported orthogo-nal vector-valued wavelets and wavelet packets,” Chaos, Solitons & Fractals. Vol.31, No.4, PP. 1024-1034,2007.

[7] S. Li, et al, “Pseudo-duals of frames with appli-cations” Appl.Comput. Harmon. Anal., 11(2), pp. 289-304, 2001.